Revêtements et monodromie de Riemann à Poincaré

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Our story starts with Gauss

Figure: Gauss

The hypergeometric equation and series, 1812

$$(1-x)\frac{d^2y}{dx^2} + (\gamma - (\alpha+\beta+1)x)\,\frac{dy}{dx} - \alpha\beta y = 0.$$

Gauss observed that the hypergeometric series

$$F(\alpha,\beta,\gamma,x) =$$

$$1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2 + \cdots$$

is a solution of the equation when γ is not a negative integer or zero (this case he excluded).

In all other cases the series is a polynomial if either $\alpha - 1$ or $\beta - 1$ is a negative integer, and otherwise converges for $x = a + bi$ by the ratio test, provided that $a^2 + b^2 < 1$.

Contiguous functions

$F(\alpha, \beta, \gamma, x)$ is contiguous to any of the six functions

$$F(\alpha \pm 1, \beta \pm 1, \gamma \pm 1, x).$$

The relationship between

$$F(\alpha, \beta, \gamma, x),\ F(\alpha + 1, \beta + 1, \gamma + 1, x), \text{ and } F(\alpha + 2, \beta + 2, \gamma + 2, x)$$

is, essentially, the hypergeometric equation.

The hypergeometric equation

'Determinatio series nostrae per Aequationem Differentialem Secundi Ordinis', (unpublished)
$P(x) = F(\alpha, \beta, \gamma, x)$ is a solution of the hypergeometric equation.

He set $1 - y = x$, and deduced that the hypergeometric equation has an independent solution

$$F(\alpha, \beta, \alpha + \beta + 1 - \gamma, 1 - x)$$

that, with the first one, forms a basis of solutions of the hypergeometric equation.

Substitutions and transformations

Gauss explicitly considered these transformations of x:

$$x \mapsto 1-x,\ \frac{1}{x},\ \frac{x}{x-1},\ \frac{x-1}{x},$$

(he omitted a composite, $x \mapsto \frac{1}{1-x}$)

and these transformations of the function:

$$x^{\mu}P,\ (1-x)^{\mu}P$$

for particular values of μ.

An equation "certainly false"

$$F(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}, y) = F(2\alpha, 2\beta, \alpha + \beta + \frac{1}{2}, 1 - y) \quad !!$$

"which equation is certainly false" (§55).

F as a function, which satisfies the hypergeometric equation, or F as the sum of an infinite series.

A function and a series

The (many-valued) ‘function’ is to be understood for all continuous changes in its fourth term, whether real or imaginary, provided the values 0 and 1 are avoided, and may have different values even though its variable has taken the same value.

The infinite series is only defined within its circle of convergence.

One would not infer from $\arcsin \frac{1}{2} = 30^\circ$ and $\sin 150^\circ = \frac{1}{2}$ that $30^\circ = 150^\circ$.

Analytic continuation

Gauss: the solutions of the differential equation exist everywhere except at 0, 1, (and ∞).
Their representation in power series is a local question, and the same function may be represented in different ways.

Gauss confronted the question of analytically continuing a function outside its circle of convergence.

Kummer's 24 solutions (1836)

In Kummer's famous table of the 24 solutions to the hypergeometric equation and their inter-relations, the variable is real.

Monodromy is a complex phenomenon,

Riemann

Figure: Riemann

Riemann's dissertation (1851)

A complex function of a complex variable is complex differentiable.
Such a function is conformal except where its derivative vanishes and it has a branch point.
Originally Gauss, 1822, Riemann deepened the insight.
The real and imaginary parts of a complex function are separately harmonic.

Domains

The independent complex variable z need not lie in $\mathbb{C}$ but in some finite domain having a boundary, spread out over $\mathbb{C}$, and covering the plane several times.

The branch points and the topology of the domain.

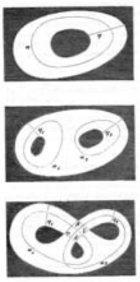

Figure: Three Riemann surfaces

Riemann – the hypergeometric equation (1857)

"The unpublished part of the Gauss's study on this series, which has been found in his *Nachlass*, was already supplemented in 1835 by the work of Kummer contained in the 15th volume of Crelle's *Journal*."
[*Göttinger Nachrichten*, Personal Report (1857)]

The hypergeometric equation as an example of linear differential equations with algebraic coefficients.

The P-function

1. It has three distinct branch points at a, b, and c, but each branch is finite at all other points;
2. A linear relation with constant coefficients exists between any three branches P, P', P'' of the function;
3. P can be written as a linear combination of two branches $P^{(\alpha)}$ and $P^{(\alpha')}$ near a, $(z-a)^{-\alpha}P^{(\alpha)}$ and $(z-a)^{-\alpha'}P^{(\alpha')}$ are single valued, and neither zero nor infinite at a. Similar conditions hold at b and c with constants β, β' and γ, γ' respectively.

None of the exponent differences $\alpha-\alpha', \beta-\beta', \gamma-\gamma'$ are integers, $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Analytic continuation around a branch point

Two linearly independent branches are continued analytically in a loop around the branch point a in the positive (anti-clockwise) direction:

$$\tilde{P}_1 = a_{11}P_1 + a_{12}P_2,$$

$$\tilde{P}_2 = a_{21}P_1 + a_{22}P_2,$$

where the a_{jk} are constants, so the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

describes what happens at a.

General analytic continuation

Let B and C be the matrices which describe the behaviour of solutions under analytic continuation around b and c respectively.

A circuit of a and b can be regarded as a circuit of c in the opposite direction, so

$$CBA = I.$$

Any closed path can be written as a product of loops around a, b, and c in the same order, or, as Riemann remarked;

"the coefficients of A, B and C completely determine the periodicity of the function".

Monodromy

The first use of the word monodromic is due to Cauchy (1851). In early 1852 he used ‘monodromique’, to denote continuous, single-valued functions.

Hermite (1851) had used a matrix to describe how an algebraic function is branched.

Riemann may be the first to have considered products of such matrices.

The term monodromy group was first used by Jordan in his *Traité* (1870), and its subsequent popularity derives from its successful use by Jordan and Klein.

The monodromy group of the hypergeometric equation

The group generated by the matrices A, B, C.
Riemann chose A, B, C to be diagonal matrices. For example, at a

$$A = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha'} \end{pmatrix}.$$

He continued the solutions near $z = a$ analytically to b and c, writing, for example

$$P^{(\alpha)} = \alpha_\beta P^{(\beta)} + \alpha_{\beta'} P^{(\beta')} \quad (1)$$

$$P^{(\alpha')} = \alpha'_\beta P^{(\beta)} + \alpha'_{\beta'} P^{(\beta')} \quad (2)$$

which introduced the matrix

$$B' = \begin{pmatrix} \alpha_\beta & \alpha_{\beta'} \\ \alpha'_\beta & \alpha'_{\beta'} \end{pmatrix}$$

and a similar matrix C' at c.

The monodromy group of the hypergeometric equation

Riemann: the coefficients of the hypergeometric equation determine the ratios of the entries in the matrices B and C, so the equation determines the monodromy.

In unpublished material he showed that the equation determines the monodromy matrices completely (using the explicit formulae in Kummer's 24 solutions, now understood as functions of a complex variable).

Conversely, he also showed that the monodromy determines the equation and, of course, when $a = 0, b = \infty, c = 1$ the differential equation for the P-function is precisely the hypergeometric equation.

Poincaré: Fuchsian and Kleinian groups

Figure: Henri Poincaré

What do you see?

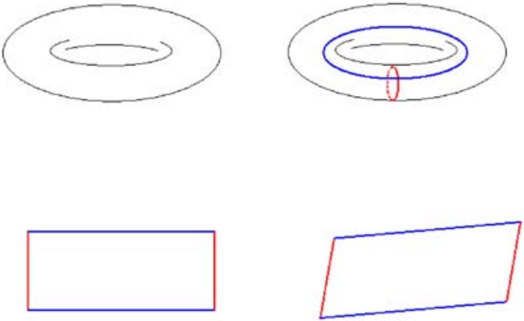

Figure: Four pictures of a torus

Not this

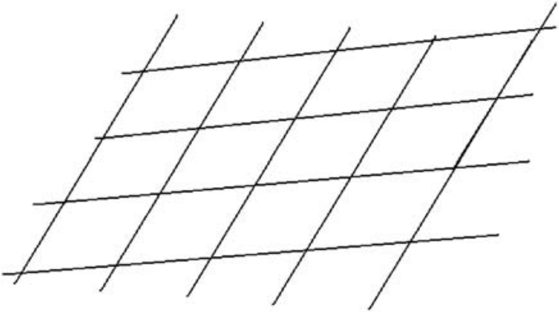

Figure: A floor tiled with parallelograms

Riemann's geometric theory of Abelian functions

Mathematicians in the 1850s, 1860s, and 1870s slowly learned to see a complex algebraic curve, $F(z, w) = 0$, as a branched covering of the z-plane or Riemann z-sphere,

and to cut it up to get a polygon with $4p$ sides identified in pairs, where p is the genus of the curve and has topological significance.

The Grand prize in mathematics of the Académie des Sciences, 1879

> *To improve in some important way the theory of linear differential equations in a single independent variable.*

(*CR*, 1879, 88, 511).
The prize was a medal to the value of 3,000 francs.
The judges: Hermite (rapporteur) Bertrand, Bonnet, Puiseux and Bouquet.

Lazarus Fuchs

Figure: Lazarus Fuchs

Poincaré

Appointed a lecturer University of Caen on 1 December 1879.

March 22, 1880, he submitted a memoir on real differential equations and their solutions; withdrawn June 14.

May 29, 1880, submitted his first account of linear ordinary differential equations in the complex domain.

[Published posthumously, in *Acta* 39 and in *Oeuvres* 1 (1928).]

Inversion

Suppose the hypergeometric equation has $f(z)$ and $g(z)$ as a basis of solutions. Poincaré considered their quotient

$$\zeta(z) = \frac{f(z)}{g(z)}.$$

Under analytic continuation ζ reproduces as

$$\zeta(z) = \frac{af(z) + bg(z)}{cf(z) + dg(z)} = \frac{a\zeta(z) + b}{c\zeta(z) + d}.$$

It is a many-valued function.
The inverse function $z = z(\zeta)$ is 'periodic':

$$z(\zeta) = z\left(\frac{a\zeta + b}{c\zeta + d}\right).$$

Tracking the image

Poincaré supposed the hypergeometric equation has real coefficients, so – before any analytic continuation – the images of the upper and lower half planes under $\zeta(z) = \frac{f(z)}{g(z)}$ form two congruent triangles, or one quadrilateral. The vertices are the images of the singular points.

The angles in the triangles are $\pi\times$ the exponent differences at the singular points.

Under analytic continuation this quadrilateral is mapped to a net of circular-arc quadrilaterals.

Poincaré asked about the extent and shape of this net.

(1) The hypergeometric equation, exponent differences $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$

z is a meromorphic single-valued function of ζ mapping a parallelogram composed of eight equilateral triangles onto the complex sphere, and $\zeta = \infty$ is its only essential singular point, so z is an elliptic function.

To illustrate his argument Poincaré added a sketch of a net of these triangles in which the hexagons are white and the two extra triangles making up the parallelogram are shaded. The shading was done incorrectly, but in what was indicative of a life-long practice Poincaré submitted it anyway.

What Poincaré failed to draw

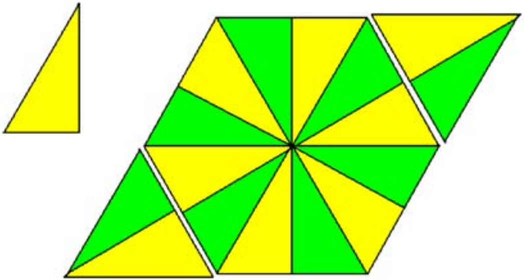

Figure: 8 triangles forming a parallelogram

(2) The hypergeometric equation, exponent differences $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{6}$.

The quadrilateral is $\alpha O \alpha' \gamma$

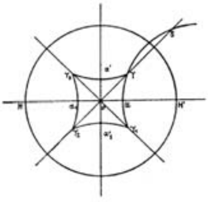

Figure: A triangle in the disc

The imags in the z-plane of the upper and lower half x-planes form a quadrilateral.
All the values of ζ – all the positions of the quadrilateral – lie inside the circle HH'.

Société de Psychologie, Paris, 1907

Depuis quinze jours, je m'efforçais de démontrer qu'il ne pouvait exister aucune fonction analogue à ce que j'ai appelé depuis les fonctions fuchsiennes; j'étais alors fort ignorant; tous les jours, je m'asseyais à ma table de travail, j'y passais une heure ou deux, j'essayais un grand nombre de combinaisons et je n'arrivais à aucun résultat. Un soir, je pris du café noir, contrairement à mon habitude, je ne pus m'endormir: les idées surgissaient en foule; je les sentais comme se heurter, jusqu'a ce que deux d'entre elles s'accrochassent, pour ainsi dire, pour former une combinaison stable. Le matin, j'avais établi 1'existence d'une classe de fonctions fuchsiennes, celles qui dérivent de la série hypergéometrique; je n'eus plus qu'a rediger les résultats, ce qui ne me prit que quelques heures.

A ce moment, je quittai Caen, où j'habitais alors, pour prendre part à une course géologique entreprise par l'École des Mines. Les péripéties du voyage me firent oublier mes travaux mathématiques ; arrivés à Coutances, nous montâmes dans un omnibus pour je ne sais quelle promenade ; au moment où je mettais le pied sur le marche-pied, 1'idée me vint, sans que rien dans mes pensées antérieures parut m'y avoir préparé, que les transformations dont j'avais fait usage pour définir les fonctions fuchsiennes étaient identiques a celles de la géometrié non-euclidienne. Je ne fis pas la vérification; je n'en aurais pas eu le temps, puisque, à peine assis dans 1'omnibus, je repris la conversation commencée, mais j'eus tout de suite une entiére certitude. De retour a Caen, je vérifiai le résultat á tête reposée pour l'acquit de ma conscience.

Non-Euclidean geometry

Poincaré was familiar with non-Euclidean geometry in which geodesics are represented as straight lines in a disc – the Beltrami model.
To study the net of triangles produced by analytic continuation, Poincaré had straightened the sides. In this picture, the angles of his triangles are not represented conformally.
It seems that he recognised his picture as a Beltrami-style picture, and then that he could convert the non-Euclidean pictures back to curvilinear, conformal pictures of non-Euclidean geometry.
The transformations are now non-Euclidean congruences, which gave him a new way to study them.

The first supplement, 28 June, 1880.

Fuchs's theorem when the exponent differences are $\rho_1 = \frac{1}{n_1}$, $\rho_2 = \frac{1}{n_2}$ and $\rho_3 = \frac{1}{n_3}$, n_j integers, and $\rho_1 + \rho_2 + \rho_3 < 1$.
The quotient ζ maps the complex z-sphere onto a quadrilateral Q.
Analytic continuation in z maps Q onto a neighbouring copy of itself obtained by rotating Q through an angle of $\frac{2\pi}{\rho_1}$ about an appropriate vertex – the geometric language is new and significant.

Poincaré's three supplements to his prize essay were omitted from Poincaré's *Oeuvres* and finally published by Scott Walter and JJG, (1997).

More rotations

Another copy of Q is obtained by a rotation through $\frac{2\pi}{\rho_3}$ about another vertex. Poincaré called these rotations M and N, and observed (1997, 31): that the copies of Q obtained by analytic continuation in this fashion fill out a disc, and that each copy of Q can be reached by a succession of crab-wise rotations

$$M^{L_1} N^{K_1} M^{L_2} N^{K_2} \dots .$$

What is a geometry?

Qu'est-ce en effet qu'une Géometrié ? C'est 1'étude du groupe d'opérations formé par les déplacements que l'on peut faire subir à une figure sans la déformer. Dans la Géometrié euclidienne ce groupe se réduit à des rotations et à des translations. Dans la pseudogéométrie de Lobatchewski il est plus compliqué.
Eh bien, le groupe des oérations combinées à 1'aide de M et de N est isomorphe à un groupe contenu dans le groupe pseudogéométrique. Étudier le groupe des opérations combinées à 1'aide de M et de N, c'est donc faire de la géométrie de Lobatchewski. La pseudogéométrie va par consequent nous fournir un langage commode pour exprimer ce que nous aurons à dire de ce groupe.

Arithmetic unexpectedly to the rescue

Je me mis alors a étudier des questions d'arithmétique sans grand résultat apparent et sans soupçonner que cela put avoir le moindre rapport avec mes recherches antérieures. Dégouté de mon insuccès, j'allai passer quelques jours au bord de la mer, et je pensai à tout autre chose. Un jour, en me promenant sur la falaise, l'idée me vint, toujours avec les mêmes caracteres de brièveté, de soudaineté et de certitude immédiate, que les transformations arithmétiques des formes quadratiques ternaires indéfinies $[x^2 + y^2 - z^2]$ *étaient identiques à celles de la géométrie non-euclidienne.*

Beyond the hypergeometric equation

Étant revenu á Caen, je réfléchis sur ce résultat, et j'en tirai les consequences; 1'exemple des formes quadratiques me montrait qu'il y avait des groupes fuchsiens autres que ceux qui correspondent à la série hypergéométrique; je vis que je pouvais leur appliquer la théorie des séries thétafuchsiennes et que, par consequent, il existait des fonctions fuchsiennes autres que celles qui dérivent de la série hypergéométrique, les seules que je connusse jus-qu'alors. Je me proposai naturellement de former toutes ces fonctions [. . .]

The Poincaré-Klein correspondence

Figure: Felix Klein

What's in a name?

Klein repeatedly objected to the name of Fuchs being used.

Poincaré admitted that "I would have chosen a different name [for the functions] had I known of Schwarz's work, but I only knew of it from your letter after the publication of my results".

He could not change the name now without insulting Fuchs.

Schwarz

Figure: Schwarz

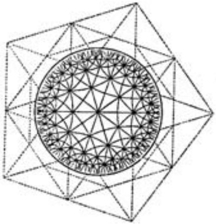

Figure: Schwarz's tessellation of the disc

Mittag-Leffler to Poincaré, 18 July 1882

When Poincaré began to publish Schwarz began to complain that Poincaré had not given him any credit.
Fuchs was full of admiration for Poincaré's beautiful discoveries but Schwarz was "almost suffocating with anger",

Poincaré wrote back to Mittag-Leffler to say that Schwarz had had the chance, and had not taken it, adding that Schwarz was really angry with himself

> *for having had an important result in his hands and not profiting from it. And I can do nothing about that.*

Poincaré and Klein

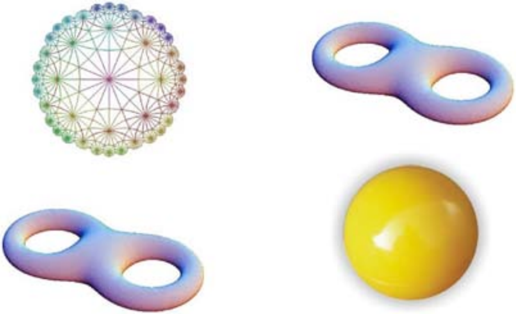

Figure: Poincaré's view (left) and Klein's (right)

Uniformisation for algebraic curves – 1

While still in Caen Poincaré published (1881):
Claim (no proof): the coordinates of points on any algebraic curve can be expressed as Fuchsian functions of an auxiliary variable.

If $F(z, w) = 0$ is the equation of an algebraic curve then there are Fuchsian functions $\varphi(x)$ and $\psi(x)$ such that $F(\varphi(x), \psi(x)) = 0$.
In other words, the curve can be parameterised by single-valued functions on the x-disc which are invariant under an appropriate Fuchsian group. T

E.g. the circle $x^2 + y^2 = 1$ with $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$.

Klein's first attempt (1882)

Claim (without proof): for each Riemann surface of genus > 1 there is a function η that maps the cut surface by analytic continuation without overlaps onto a $2p$-connected region of the sphere.

Only Klein invoked the birational classification of surfaces and so established a parameter count that made it plausible that a correspondence between discontinuous groups and curves might work.

Klein's 'Neue Beiträge'

The analytic continuation of η moves the image of the cut surface around on the sphere. $2p$ cuts make a surface of genus p simply connected, when it becomes a polygon of $4p$ sides. There are $2p$ lengths and $4p$ angles in such a polygon, so $6p - 3$ independent real coordinates, but in the upper half plane model, η may be replaced by $\frac{\alpha\eta+\beta}{\gamma\eta+\beta}$ because the first edge can be put any where, so 3 parameters are inessential, and the function η depends on $6p - 6$ real or $3p - 3$ complex parameters.

The space of moduli for Riemann surfaces also has complex dimension $3p - 3$ and so Klein made the audacious claim that every Riemann surface corresponds according to a unique group.

Klein's Easter holiday, Nordeney 1882

I only stayed there for eight days because the life was too miserable, since violent storms made any excursions impossible and I had severe asthma [. . .] On the last night, 22nd to 23rd March, when I needed to sit on the sofa because of my asthma, suddenly the 'Grenzkreis theorem' appeared before me at about two thirty as it was already quite properly prefigured in the picture of the 14*-gon in volume 14 of the Mathematische Annalen. On the following morning [. . .] I knew that I had a great theorem. Arrived [home] I wrote all it at once, dated it the 27th of March, sent it to Teubner and allowing for corrections to Poincaré and Schwarz, and for example to Hurwitz.*

The 14-gon

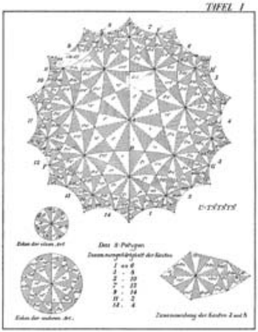

Figure: The tessellation for a particular Riemann surface of genus 3

The uniformisation theorem; Poincaré 1883

A proof sketch that extended it to all many-valued analytic functions, not just the algebraic functions.

Claim: if y is a many-valued analytic function of x given by an equation of the form $f(x, y) = 0$ then x and y can be expressed as single-valued functions of a complex variable z.

See (H.P. de Saint-Gervais, *Uniformisation des surfaces de Riemann*, 2010).

Poincaré's first attempt at a proof, 1883

While the paper was in press in *Acta mathematica* Mittag-Leffler said :

"Isn't that terrific? In analysis there is no theorem which in its striking simplicity surpasses this." (Stubhaug 2010, 292).

The domain – analysis

A domain for the variable z.
In the case of the cubic curves the z-domain is the complex plane.

For algebraic curves of genus > 1 Poincaré believed it would be the non-Euclidean disc.

In each of these cases there is, or seems to be, a fundamental domain – a suitable polygon – copies of which fill out the requisite domain.
Corresponding points in different copies of the fundamental polygon can be joined by paths, and these paths map down to closed paths on the surface defined by $f(x, y) = 0$.

Lifting paths on a torus

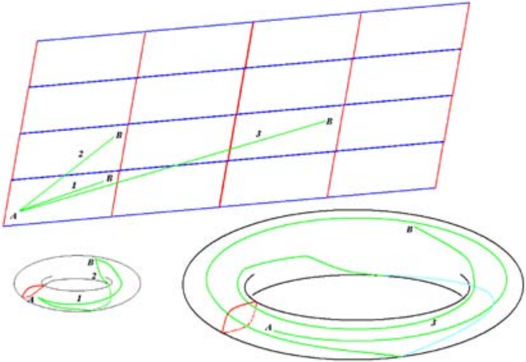

Figure: Three paths from A to B

Poincaré therefore considered running this process in reverse:
opening out all the closed paths from a fixed but arbitrary starting point P on $f(x, y) = 0$,
letting their endpoints collectively define the sought-for z-domain.

A path on the surface will consist of points $(x(t), y(t))$, where t is a real variable and $P = (x(0), y(0))$. Because y is a many-valued function of x it can happen that when $x(t_0) = x(0), t_0 > 0$, nonetheless $y(t_0) \neq y(0)$.

The domain – synthesis

Open out all the closed paths on $f(x, y) = 0$ from a fixed but arbitrary starting point $P = (x(0), y(0))$.
Let their endpoints collectively define the sought-for z-domain.
Define a map from this domain to the curve

$$z \mapsto (p(z), q(z)), \text{ where } f(p(z), q(z)) = 0.$$

Prove this map is analytic.

Uniformisation

Poincaré's argument was not convincing, or even clear.
In 1900, in his Paris address on the mathematical problems facing the 20th century, Hilbert stressed that it was extremely desirable to check that the uniformising map was in particular surjective – no points on the curve are missed.
This was not clear in Poincaré's original argument.

The theorem was first proved by Poincaré and Koebe independently in 1907.

Motions in the covering space

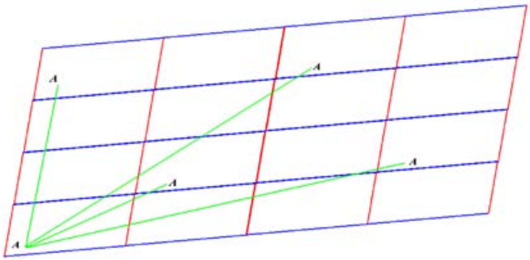

Figure: The lifts of of four loops – or four translations of a parallelogram lattice

The group action

Poincaré and Klein claimed that every Riemann surface (genus $p > 1$) is obtained from the non-Euclidean disc from the action of a group that moves a fundamental region around.

The disc is the universal covering space for the Riemann surface.

The group is also the fundamental group of the surface – its elements are [homotopy classes of] loops on the surface.

Different groups, different polygons

Figure: Larger figures, smaller groups

The group moving a polygon around en bloc is a normal subgroup of the group moving, a sub-polygon around en bloc.
The big polygon and the sub-polygon each give rise to a Riemann surface.

Four squares in a square

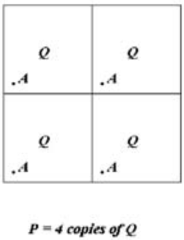

Figure: paths AA are loops for Q but not for P

When Q generates the Riemann surface, paths AA correspond to loops. When P generates the Riemann surface, paths AA do not correspond to loops.

Fields

A (nonconstant) analytic map σ from a Riemann surface X to another Y is a finite branched covering of Y by X.
Every Riemann surface has a field of meromorphic functions upon it: $M(X)$ and $M(Y)$.
The map σ yields a map

$$\sigma^* : M(Y) \to M(X).$$

Conclusion

P and Q are polygons, P is made of copies of Q.
X and Y are the corresponding Riemann surfaces.
There is a covering map, $\sigma : X \to Y$
– distinct points in X correspond to the same point in Y.

Their fundamental groups are $\pi_1(X)$ and $\pi_1(Y)$.

$$\pi_1(X) \lhd \pi_1(Y).$$

$$\sigma^* : M(Y) \to M(X).$$

This is a Galois correspondence.