Dedekind, Frobenius and the beginning of representation theory: cooperation and conflicting views

Leo Corry – Tel Aviv University

Dedekind, Frobenius

Google Search

I’m Feeling Lucky

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Dedekind

Richard Dedekind

Dedekind cuts

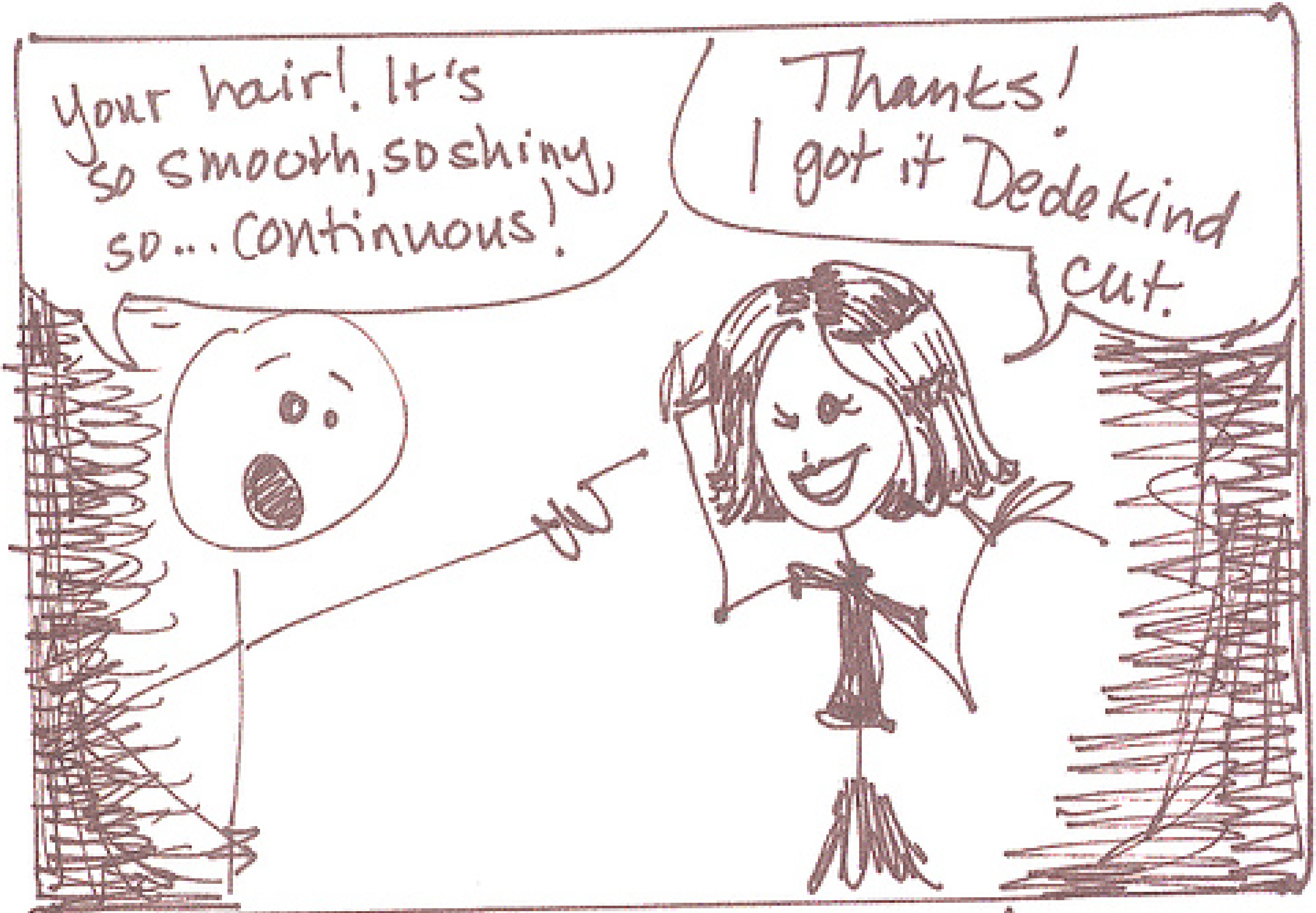
Dedekind ideals

Dedekind domains

Dedekind numbers

Dedekind eta function

Dedekind's contributions to the foundations of mathematics



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Frobenius

Georg Ferdinand Frobenius

Frobenius algebra

Frobenius theorem (differential geometry)

Frobenius theorem (real division algebras)

Frobenius representation

Frobenius substitution

Frobenius norm

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Frobenius

Frobenius substitution

Frobenius’s most famous find …

… There is, however, no evidence that Frobenius perpetrated the substitution …

Frobenius's most famous find was the brass head known as Olokun … He bought it for six pounds and a bottle of whisky. … Underwood and Fagg have demonstrated that the head purporting to be this one, … and which is now in the Ife museum is not an original Ife head, but a copy made by sandcasting …

Leo Frobenius (1873- 1938)

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Leo Frobenius (1873- 1938)


Dedekind, Frobenius and the beginning of representation theory

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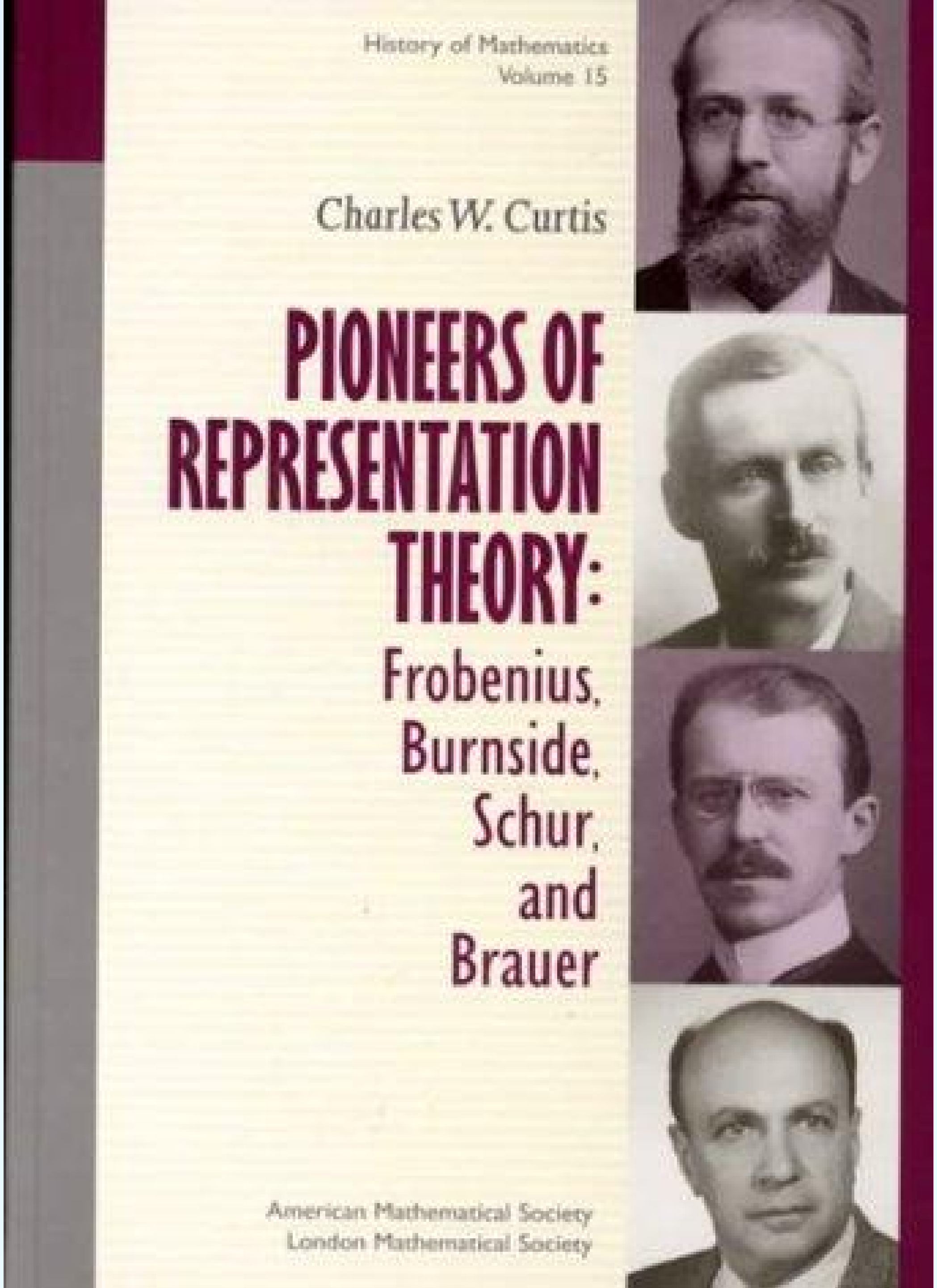
Sources and Studies
in the History of Mathematics and
Physical Sciences

THOMAS HAWKINS

EMERGENCE OF THE THEORY OF LIE GROUPS

An Essay in the History of Mathematics 1869–1926

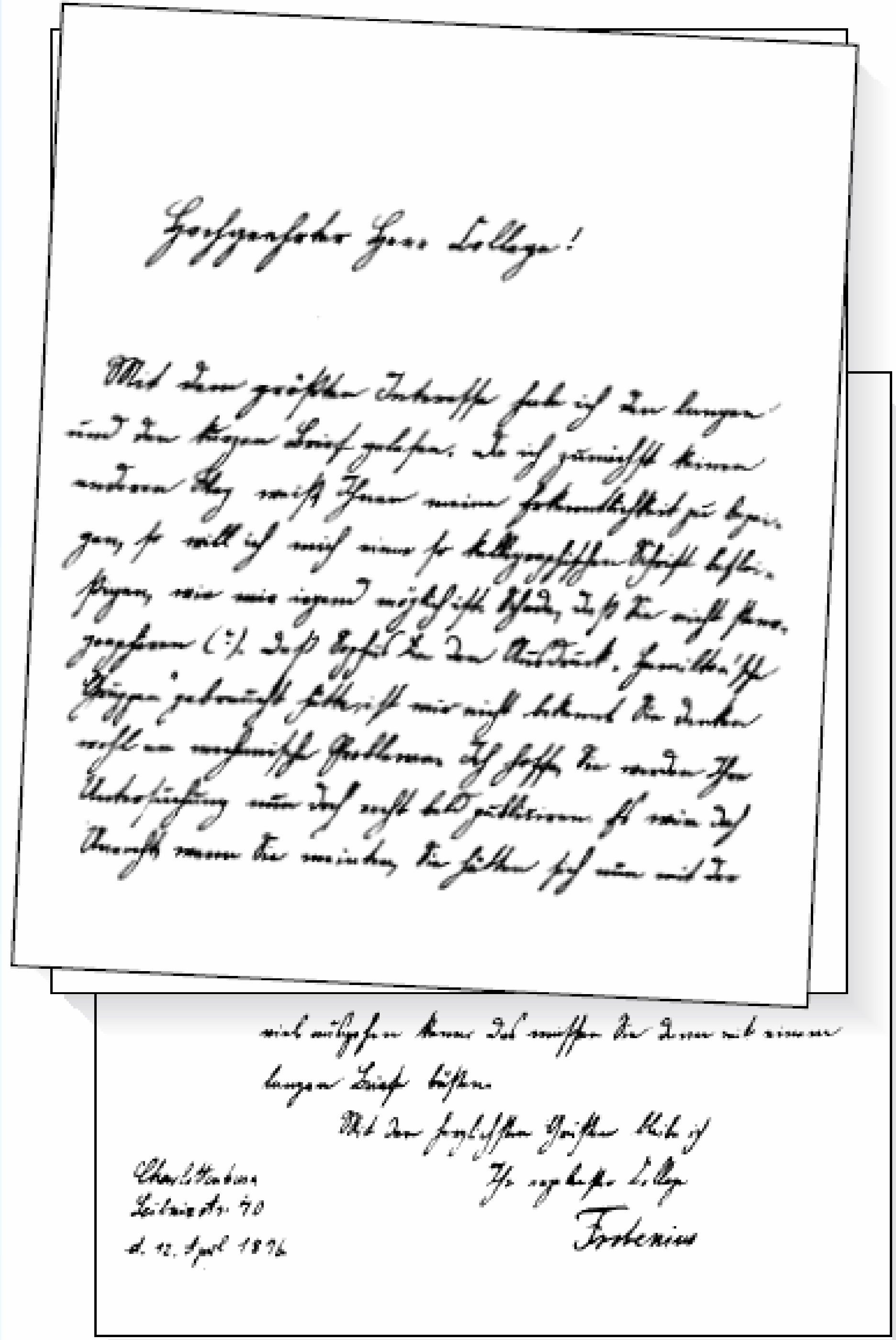
Springer


History of Mathematics
Volume 15
Charles W. Curtis
PIONEERS OF REPRESENTATION THEORY:
Frobenius, Burnside, Schur, and Brauer
American Mathematical Society
London Mathematical Society

Y.T. Lam, "Representations of Finite Groups: A Hundred Years", *Notices AMS* (1998).

Keith Conrad, "The Origins of Representation Theory", Preprint.

Ralf Haubrich (ed.):
The Dedekind-Frobenius Correspondence (1881-1896) [forthcoming?]


Hochgeehrter Herr College!

Charlottenburg
Leibnizstr. 70
d. 12. April 1896

Frobenius

Frédéric Brechenmacher

“La controverse de 1874 entre Camille Jordan et Leopold Kronecker”, *Revue d’Histoire des Mathématiques*, 13 (2007).

“Une histoire de l’universalité des matrices mathématiques”, *Revue de Synthèse,* 131 (2010).

Dedekind, Frobenius and the beginning of representation theory:

Dedekind's letter to Frobenius, March 25, 1896

Dedekind to Frobenius (March 25, 1896):

On the whole, one may well suppose that the properties of a group *G* regarding its sub-groups will be reflected in the decomposition of its determinant θ. However, except for a trace, which indicates a connection between the number of ordinary linear factors of θ and the normal sub-groups *A* of *G* …, *I have found nothing at* all; and actually it is quite likely that for the present, little will come out of the whole thing …

Dedekind's "Determinant of *G*"

Given G – a group of order *n*: $\{g_1, g_2, \ldots g_n\}$ $[g_1 = e]$

Associate a variable x_i with each g_i (and x_i' with g_i^{-1})

Define the "determinant" θ of G, a homogeneous polynomial of degree *n*, in *n* variables

$$\theta\,(x_1, x_2, \ldots x_n) = \begin{vmatrix} x_{11'} & x_{21'} & \ldots & x_{n1'} \\ x_{21'} & x_{22'} & \ldots & x_{2n'} \\ \ldots & & & \\ x_{n1'} & x_{n2'} & \ldots & x_{nn'} \end{vmatrix}$$

Dedekind to Frobenius (March 25, 1896):

… a connection between the number of ordinary linear factors of θ and the normal sub-groups A of G …

If G is abelian:

θ factors into exactly n linear forms over $\mathbb{C}$

" … a theorem, which in this generality, as I believe, has not been announced as yet".

Dedekind to Frobenius (March 25, 1896):

… But if G is not an abelian group, then its determinant θ possesses, as far as I have checked, besides linear factors … also factors of higher

And a non-proven conjecture: in the non-abelian case, the number of linear factors equals the index of the commutator sub-group of *G*.

In the background to Dedekind’s musing with these ideas:

•Gauss on characters of finite abelian groups (assigning numerical properties to classes of binary quadratic forms)

•Higher reciprocity and the Legendre symbol

Dedekind’s recent work on number theory (Dirichlet’s *Vorlesungen*)

Dedekind's own work on algebraic number fields and the group determinant:

K – a normal extension of *Q*

G: $\{\pi_1, \pi_2, \ldots \pi_n\}$ - the Galois group of *K*

Given $\omega_1, \omega_2, \ldots \omega_n$ linearly independent in *K*, then the discriminant of K is defined as:

$$\begin{vmatrix} \omega_1\pi_1 & \omega_2\pi_1 & \ldots & \omega_n\pi_1 \\ \omega_1\pi_2 & \omega_2\pi_2 & \ldots & \omega_n\pi_2 \\ \ldots & & & \\ \omega_1\pi_n & \omega_2\pi_n & \ldots & \omega_n\pi_n \end{vmatrix}$$

Dedekind's own work on algebraic number fields and the group determinant:

In particular, if $\omega_1, \omega_2, \ldots \omega_n$ are taken to be the collection of an element ω in K and all of its conjugates $(\omega\pi_1, \omega\pi_2, \ldots \omega\pi_n)$, then the discriminant of K is:

$$\begin{vmatrix} \omega_1\pi_1 & \omega_2\pi_1 & \ldots & \omega_n\pi_1 \\ \omega_1\pi_2 & \omega_2\pi_2 & \ldots & \omega_n\pi_2 \\ \ldots & & & \\ \omega_1\pi_n & \omega_2\pi_n & \ldots & \omega_n\pi_n \end{vmatrix} \Rightarrow \begin{vmatrix} \omega\pi_1\pi_1 & \omega\pi_2\pi_1 & \ldots & \omega\pi_n\pi_1 \\ \omega\pi_1\pi_2 & \omega\pi_2\pi_2 & \ldots & \omega\pi_n\pi_2 \\ \ldots & & & \\ \omega\pi_1\pi_n & \omega\pi_2\pi_n & \ldots & \omega\pi_{\pi n}\pi_n \end{vmatrix}$$

Dedekind’s own work on algebraic number fields and the group determinant:

The discriminant of K $\Rightarrow$ the group determinant

$$\begin{vmatrix} \pi_1\pi_1 & \pi_2\pi_1 & \dots & \pi_n\pi_1 \\ \pi_1\pi_2 & \pi_2\pi_2 & \dots & \pi_n\pi_2 \\ \dots & & & \\ \pi_1\pi_n & \pi_2\pi_n & \dots & \pi_{\pi n}\pi_n \end{vmatrix} \Leftarrow \begin{vmatrix} \omega\pi_1\pi_1 & \omega\pi_2\pi_1 & \dots & \omega\pi_n\pi_1 \\ \omega\pi_1\pi_2 & \omega\pi_2\pi_2 & \dots & \omega\pi_n\pi_2 \\ \dots & & & \\ \omega\pi_1\pi_n & \omega\pi_2\pi_n & \dots & \omega\pi_{\pi n}\pi_n \end{vmatrix}$$

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$$\begin{vmatrix} \omega\pi_1\pi_1 & \omega\pi_2\pi_1 & \dots & \omega\pi_n\pi_1 \\ \omega\pi_1\pi_2 & \omega\pi_2\pi_2 & \dots & \omega\pi_n\pi_2 \\ \dots & & & \\ \omega\pi_1\pi_n & \omega\pi_2\pi_n & \dots & \omega\pi_n\pi_n \end{vmatrix}$$

$$\begin{vmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \dots & & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}$$

Variables x_{ij} instead of numbers

A polynomial instead of a value

Dedekind's own work on algebraic number fields and the group determinant:

The discriminant of K $\Rightarrow$ the group determinant

$$\begin{vmatrix} \pi_1\pi_1 & \pi_2\pi_1 & \ldots & \pi_n\pi_1 \\ \pi_1\pi_2 & \pi_2\pi_2 & \ldots & \pi_n\pi_2 \\ \ldots & & & \\ \pi_1\pi_n & \pi_2\pi_n & \ldots & \pi_{\pi n}\pi_n \end{vmatrix} \Leftarrow \begin{vmatrix} \omega\pi_1\pi_1 & \omega\pi_2\pi_1 & \ldots & \omega\pi_n\pi_1 \\ \omega\pi_1\pi_2 & \omega\pi_2\pi_2 & \ldots & \omega\pi_n\pi_2 \\ \ldots & & & \\ \omega\pi_1\pi_n & \omega\pi_2\pi_n & \ldots & \omega\pi_{\pi n}\pi_n \end{vmatrix}$$

x_e

$$\begin{vmatrix} x_{11'} & x_{21'} & \ldots & x_{n1'} \\ x_{21'} & x_{22'} & \ldots & x_{2n'} \\ \ldots & & & \\ x_{n1'} & x_{n2'} & \ldots & x_{nn'} \end{vmatrix} \Leftarrow \begin{vmatrix} x_{11} & x_{21} & \ldots & x_{n1} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \ldots & & & \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{vmatrix}$$

Dedekind to Frobenius (April , 1896):

"In case you still want to deal with the group determinant, I allow myself to send you two examples that I thoroughly calculated on February 1886 … "

1896: Frobenius Takes the Challenge

- Defined characters of general finite groups
- Proved the main theorems about them
- Applied his new theory to solve the problem of factoring the determinant of a general group into irreducible factors

Determinant, characters and representations: A sketch of their relationship

Let H = $\{ \sum_{g \in G} \alpha_g g$, with α_g in $\mathbb{C}$ $\}$

H is the group algebra of G, i.e., the g-dimensional linear associative algebra over $\mathbb{C}$.

Consider the linear transformation T_g :

$$T_g\left(\sum_{g' \in G} \alpha_{g'} g'\right) = \sum_{g' \in G} \alpha_{g'} g' g.$$

Determinant, characters and representations:

A sketch of their relationship

Let $\sigma(g)$ be the matrix representation of T_g with respect to the basis g_1^{-1}, g_2^{-1} , … , g_n^{-1} . Then $g \rightarrow \sigma(g)$ is the right regular representation of G, and

$$\theta = \det [x_1\sigma(g_1)+x_2\sigma(g_2)+ \ldots + x_n\sigma(g_n)]$$

Determinant, characters and representations:
A sketch of their relationship

Now, if M is a non-singular $n*n$ matrix over $\mathbb{C}$, such that for all g in G,

$$M\sigma(g)M^{-1} = \begin{pmatrix} \mu(g) & 0 \\ 0 & \nu(g) \end{pmatrix},$$

$\mu(g)$ and $\nu(g)$ being matrices, $r*r$ and $s*s$, respectively, then

$$\theta(x_1, x_2, \dots x_n) = \Phi(x_1, x_2, \dots x_n) * \Psi(x_1, x_2, \dots x_n)$$

with Φ and Ψ - polynomials of degrees r,s

Determinant, characters and representations:
A sketch of their relationship

$$\theta\,(x_1, x_2, \ldots x_n) = \Phi(x_1, x_2, \ldots x_n) * \Psi(x_1, x_2, \ldots x_n)$$

with Φ and Ψ - polynomials of degrees r,s

Hence: the decomposition of regular representations into irreducible representations is equivalent to the decomposition of the group determinant into irreducible factors, with corresponding degrees.

Determinant, characters and representations: A sketch of their relationship

Frobenius' own formulation of the problem:

If $$\theta = \prod_{\lambda=1}^{l} [\Phi_\lambda(x)]^{e_\lambda}$$

is the factorization of the determinant into different irreducible factors of Φ_λ of degree f_λ, then, how does this factorization reflect the properties of the group G?

Determinant, characters and representations: A sketch of their relationship

Frobenius' own formulation of the problem:

If $\theta = \prod_{\lambda=1}^{l} [\Phi_\lambda(x)]^{e_\lambda}$ is the factorization of the determinant into different irreducible factors of Φ_λ of degree f_λ, then, how does this factorization reflect the properties of the group G?

- Is the number of linear factors as conjectured by Dedekind?
- How are l, e_λ, f_λ, related to G?

Determinant, characters and representations: A sketch of their relationship

$$\theta = \prod_{\lambda=1}^{l} [\Phi_{\lambda}(x)]^{e_{\lambda}}$$

In his 1896 articles, Frobenius investigated symmetric matrices and applied their properties to $\sigma(g)$ in order to answer the above questions. Among other things:

- Dedekind's conjecture was correct
- l equals the number of conjugate classes of G
- $e_{\lambda} = f_{\lambda}$ i.e., each factor Φ_{λ} occurs as often as its degree
- A generalization of the concept of character
- Fundamental results about representations of groups

In the background to Frobenius' interest in, and reaction to, Dedekind's suggestion:

- The theory of linear differential operators
- Linear forms with integer coefficients
- Improved proofs of Sylow's theorem
- Linear and bilinear forms
- The theory of biquadratic forms

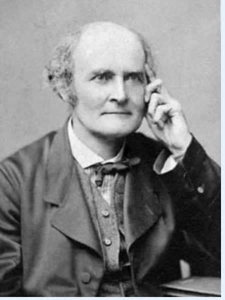
In the background to Dedekind’s suggestion to Frobenius:

- •Gauss on characters of finite abelian groups (assigning numerical properties to classes of binary quadratic forms)
- •Higher reciprocity and the Legendre symbol
- •Dirichlet’s application of analytical methods to number theory
- •Recent work on hypercomplex systems (after Hamilton)
- Dedekind recent work on number theory (Dirichlet’s Vorlesungen)

Determinant, characters and representations:

Why? Why in this way? Why at this point?

- What is “group theory” in 1896?
- What is “algebra” in 1896?
- What is “group theory” for Frobenius? (or for Dedekind?)
- What is “algebra” for Dedekind (or for Frobenius?)
- How is all of this related with the work of Galois?



Determinant, characters and representations:

Why? Why in this way? Why at this point?

Groups as groups of permutations (substitutions)

Cayley (1878): “The Theory of Groups”, *AJM* Vol. 1.

The general problem of finding all the groups of an order *n* is really identical with the apparently less general problem of finding all the groups of the same order *n*, that can be formed with the substitution upon *n* letters … This, however, in any wise shows that the best or the easiest way of treating the general problem is thus to regard it as a problem of substitutions: and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions.

Determinant, characters and representations:

Why? Why in this way? Why at this point?

Groups of permutations as groups of linear substitutions

Weber (1896): *Lehrbuch der Algebra*, Vol. 2.

The importance in algebra of linear substitutions and in particular of the finite groups that they define concerns the fact that groups of permutations of *n* elements can be represented as groups of linear representations.

Determinant, characters and representations: Why? Why in this way? Why at this point?

Groups of permutations as groups of linear substitutions

Burnside, W. (1897), Theory of Groups Of Finite Order, Cambridge.

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications aside, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with the properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

- **What is “group theory” in 1896?**
- **What is “algebra” in 1896?**
- **How is all of this related with the work of Galois?**

B.L. van der Waerden (1985):
A History of Algebra-From al-Kharizmi to Emmy Noether

Modern algebra begins with Evariste Galois. With Galois, the character of algebra changed radically. Before Galois, the efforts of algebraists were mainly directed towards the solution of algebraic equations... After Galois, the efforts of the leading algebraists were mainly directed towards the structure of rings, fields, algebras, and the like.

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- **What is “algebra” in 1896?**
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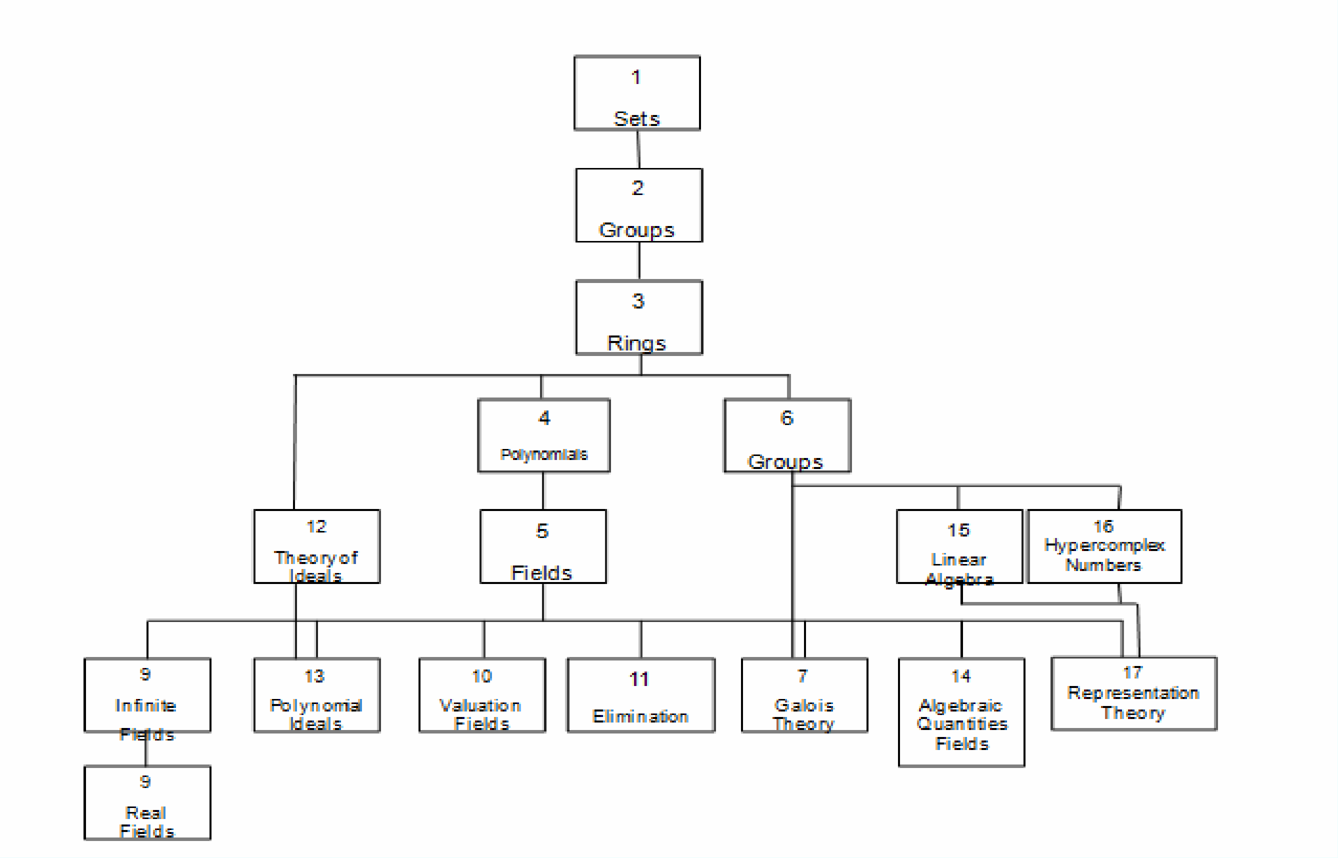
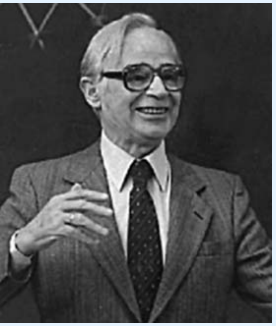
B.L. van der Waerden (1930): *Moderne Algebra*





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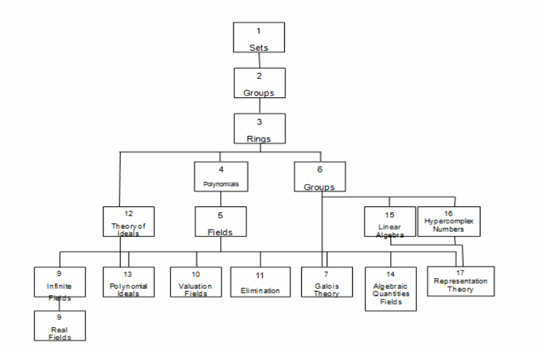
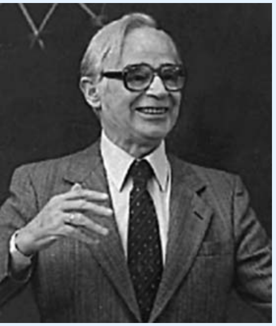




With Galois, the character of algebra changed radically.... After Galois, the efforts of the leading algebraists were mainly directed towards the structure of rings, fields, algebras, and the like.

With Galois, a long and complex process started, which *eventually* led to a radical change in the character of algebra.... the efforts of the leading algebraists were *increasingly* directed towards the structure of rings, fields, algebras, and the like (but there were other things going around). *Finally* (with Noether – and with VDW’s book), the idea that algebra deals with structures was consolidated, and it became very influential on subsequent developments.

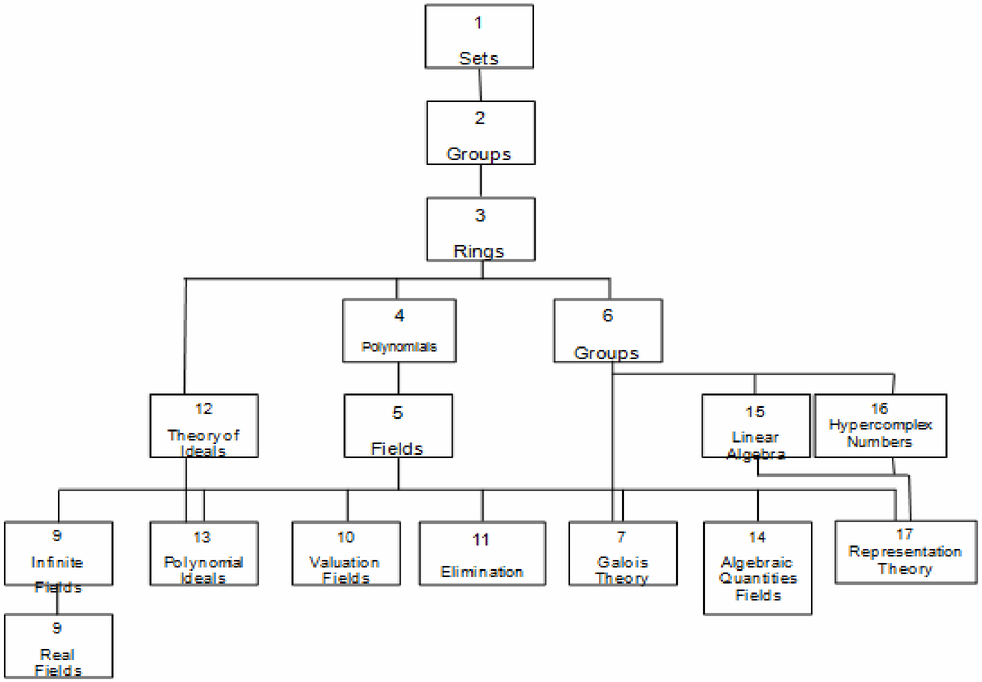
Abstract Theory of Fields

Ernst Steinitz

“Algebraische Theorie der Körper“, *Crelle* 137 (1910).

Abstract Theory of Rings

Es steht alles schon bei Dedekind?



- What is “group theory” Dedekind?
- What is “algebra” for Dedekind Frobenius?
- How is all of this related with the work of Galois?

Lectures on Galois Theory (1856-57)

Groups of Substitutions

The properties of these groups are thoroughly discussed in advance

These are innovative, efficient tools for discussing the question of the roots (in the rational domains) of polynomial equations

Rational Domains

Their properties are briefly discussed when needed but never systematically analyzed.

A subset of the complex numbers closed under the four arithmetic operations

These systems and their interrelations constitute the subject-matter of “higher algebra”

- What is “group theory” Dedekind?
- What is “algebra” for Dedekind Frobenius?
- How is all of this related with the work of Galois?

Theory of Algebraic Number Fields (1871, 1877, 1879, 1894)

Ideals, Modules

Fields, Algebraic Integers

These are innovative, efficient tools for discussing the question of unique factorization

The basic Concepts, theorems and proofs are successively formulated so as to avoid the need for choosing specific elements

Frobenius to Weber (1983):

Your announcement of a work on algebra makes me very happy... Hopefully you will follow Dedekind's way, yet avoid the highly abstract approach that he so eagerly pursues now. His newest edition (of the *Vorlesungen) contains so many* beauty ideas, ... but his permutations are too flimsy, and it is indeed unnecessary to push the abstraction so far. I am therefore satisfied, that you write the Algebra and not our venerable friend and master, who had also once considered that plan.

Determinant, characters and representations:

Why? Why in this way? Why at this point?

In the background to Frobenius' interest in, Dedekind's suggestion:

- The theory of linear differential operators
- Linear forms with integer coefficients
- Improved proofs of Sylow's theorem
- Linear and bilinear forms
- The theory of biquadratic forms

Matrices?

Noether and Matrices:

Paul Dubreil, (1983) Souvenirs d'un boursier Rockefeller 1929-1931. *Cahiers du séminaire d'histoire des mathématiques.*

"Le cours d'Emmy Noether n'était pas facile à suivre … J'ai eu un jour une difficulté à propos d'une affirmation qui ne ma paraissait pas justifiée ni dans son cours, ni dans son mémoire: une démonstration s'obtenait sans peine par un calcul de matrices mais j'étais devenu assez "noetherien" pour ne pas m'en satisfaire. A la fin de la leçon suivante, je suis allé poser la question à Emmy Noether qui repoussa énergiquement les matrices et, après trois secondes de réflexion, me montra combien la chose était claire si l'on jonglait adroitement avec les modules !

Dedekind, Frobenius and the beginning of representation theory: cooperation and conflicting views