Galois theory in Göttingen (Noether, Artin. . . .)

Artin: "Since my mathematical youth I have been under the spell of the classical theory of Galois. This charm has forced me to return to it again and again, and to try to find new ways to prove its fundamental theorems."

I was a witness how Artin gradually developed his best known simplification, his proof of the main theorem of Galois Theory.

I was a witness how Artin gradually developed his best known simplification, his proof of the main theorem of Galois Theory.

The situation in the thirties was determined by the existence of an already well developed algebraic theory initiated by one of the most fiery spirits that ever invented mathematics, the spirit of Galois.

Zassenhaus

Kiernan:

“ARTIN took a revolutionary new look at the theory, and took up the concept stated implicitly by GALOIS and announced, unheard, by DEDEKIND and WEBER: The theory is concerned with the relation between field extensions and their groups of automorphisms.”

Weber 1912:

"Primarily two great general concepts lead to a mastery of modern algebra . . . the concepts of *group* and *field*"

Weber 1912:

"Primarily two great general concepts lead to a mastery of modern algebra . . . the concepts of *group* and *field*"

No general account of Galois theory between Weber and Artin emphasized this.

Weber 1912:

"Primarily two great general concepts lead to a mastery of modern algebra . . . the concepts of *group* and *field*"

No general account of Galois theory between Weber and Artin emphasized this. But progress was not only in general accounts.

Weber 1912:

"Primarily two great general concepts lead to a mastery of modern algebra ... the concepts of *group* and *field*"

No general account of Galois theory between Weber and Artin emphasized this. But progress was not only in general accounts.

Kiernan: "In the early decades of the 20th century, German mathematicians such as EMMA NOETHER (1882–1935) began to examine in detail fields and their generalizations."

So, what did Artin do for Galois theory?

So, what did Artin do for Galois theory?

Generalized Dedekind's fundamental theorem of Galois theory from algebraic subfields of the complex to arbitrary fields (esp. considering separability).

So, what did Artin do for Galois theory?

Generalized Dedekind's fundamental theorem of Galois theory from algebraic subfields of the complex to arbitrary fields (esp. considering separability).

Eliminated primitive elements from the proof.

So, what did Artin do for Galois theory?

Generalized Dedekind's fundamental theorem of Galois theory from algebraic subfields of the complex to arbitrary fields (esp. considering separability).

Eliminated primitive elements from the proof.

Everyone today uses Artin's Fundamental theorem.

So, what did Artin do for Galois theory?

Generalized Dedekind's fundamental theorem of Galois theory from algebraic subfields of the complex to arbitrary fields (esp. considering separability).

Eliminated primitive elements from the proof.

Everyone today uses Artin's Fundamental theorem.
Many prove it with primitive elements.

Artin made Galois theory
into

Moderne Algebra.

He built it on degree and dimension rather than calculation.

He built it on degree and dimension rather than calculation.

Besides understating the substance of Artin's work, Kiernan misses its place in the future of Galois theory – esp. class field theory and cohomology.

Noether thoroughly absorbed and advanced Dedekind-Weber on fields and groups.

Noether thoroughly absorbed and advanced Dedekind-Weber on fields and groups.

E.g. a sublime triviality: Any field K acted on by any finite group G is Galois over the fixed field K^G.

Noether thoroughly absorbed and advanced Dedekind-Weber on fields and groups.

E.g. a sublime triviality: Any field K acted on by any finite group G is Galois over the fixed field K^G.

She applies this triviality, plus her own group invariant theory, to the inverse Galois problem.

Let finite group G permute the field $K = \mathbb{Q}(x_e, x_g, \ldots, x_h)$ generated by elements of G.

Let finite group G permute the field $K = \mathbb{Q}(x_e, x_g, \ldots, x_h)$ generated by elements of G.

Noether showed K^G is f.g. over $\mathbb{Q}$, and if the generating set can be shrunk to the size of G then K^G is a polynomial ring over $\mathbb{Q}$.

Let finite group G permute the field $K = \mathbb{Q}(x_e, x_g, \ldots, x_h)$ generated by elements of G.

Noether showed K^G is f.g. over $\mathbb{Q}$, and if the generating set can be shrunk to the size of G then K^G is a polynomial ring over $\mathbb{Q}$.

K is Galois over K^G.

Let finite group G permute the field $K = \mathbb{Q}(x_e, x_g, \ldots, x_h)$ generated by elements of G.

Noether showed K^G is f.g. over $\mathbb{Q}$, and if the generating set can be shrunk to the size of G then K^G is a polynomial ring over $\mathbb{Q}$.

K is Galois over K^G.

By Hilbert irreducibility G is a Galois group over $\mathbb{Q}$.

Most importantly for Artin and the future of Galois theory, Noether created algebra up to isomorphism – not in the general-logical sense of the model theorists or ‘structuralists’

Most importantly for Artin and the future of Galois theory, Noether created algebra up to isomorphism – not in the general-logical sense of the model theorists or 'structuralists'

– but in the specific mathematical sense she called "purely set-theoretic" and "independent of any operations."

Inclusions of submodules, and induced maps between quotients, replace elements and equations.

Inclusions of submodules, and induced maps between quotients, replace elements and equations.

Noether's *homomorphism and isomorphism* theorems.

Inclusions of submodules, and induced maps between quotients, replace elements and equations.

Noether's *homomorphism and isomorphism* theorems.

What today is done by *exact sequences*.

Inclusions of submodules, and induced maps between quotients, replace elements and equations.

Noether's *homomorphism and isomorphism* theorems.

What today is done by *exact sequences*.

As Artin works over fields, all this turns into dimension of vector spaces.

Independance of characters gives a lower bound on dimension of certain vector spaces.

Independance of characters gives a lower bound on dimension of certain vector spaces.

Lower bounds give other upper bounds –

Independance of characters gives a lower bound on dimension of certain vector spaces.

Lower bounds give other upper bounds – if one space were too large, it would provide solutions to linear equations reducing another space too much.

Independance of characters gives a lower bound on dimension of certain vector spaces.

Lower bounds give other upper bounds – if one space were too large, it would provide solutions to linear equations reducing another space too much.

Artin proves, not equations between elements, but equations between orders of groups, indices of normal subgroups, and dimensions of spaces.

Noether, with Bauer and Hasse, was already using crossed products and representation modules for class field theory.

Noether, with Bauer and Hasse, was already using crossed products and representation modules for class field theory.

Most famously Noether's Hilbert's Theorem 90.

Noether, with Bauer and Hasse, was already using crossed products and representation modules for class field theory.

Most famously Noether's Hilbert's Theorem 90.

She was consciously bringing Galois theory into her framework of rings, ideals,and modules

Noether, with Bauer and Hasse, was already using crossed products and representation modules for class field theory.

Most famously Noether's Hilbert's Theorem 90.

She was consciously bringing Galois theory into her framework of rings, ideals,and modules – on her 'purely set theoretic foundations'.

Noether, with Bauer and Hasse, was already using crossed products and representation modules for class field theory.

Most famously Noether's Hilbert's Theorem 90.

She was consciously bringing Galois theory into her framework of rings, ideals,and modules – on her 'purely set theoretic foundations'.

We will look back, and then forward.

To understand the 1930s, we must appreciate how:

Es steht alles schon bei Dedekind.

To understand the 1930s, we must appreciate how:

Es steht alles schon bei Dedekind.

Dedekind makes the crucial observation that algebraic independence of a is linear independence of its powers $a, a^2, \ldots a^n, \ldots$.

Dedekind §161 proves

For any system $\phi_1, \phi_2, \ldots, \phi_n$ of n permutations of a field K, infinitely many numbers in K have n distinct images under Φ.

Dedekind §161 proves

For any system $\phi_1, \phi_2, \ldots, \phi_n$ of n permutations of a field K, infinitely many numbers in K have n distinct images under Φ.

Dedekind defines characters of finite Abelian groups

Dedekind §161 proves

For any system $\phi_1, \phi_2, \ldots, \phi_n$ of n permutations of a field K, infinitely many numbers in K have n distinct images under Φ.

Dedekind defines characters of finite Abelian groups

Artin extends to all groups so the above is a case of independence of characters.

Dedekind §161 proves

For any system $\phi_1, \phi_2, \ldots, \phi_n$ of n permutations of a field K, infinitely many numbers in K have n distinct images under Φ.

Dedekind defines characters of finite Abelian groups

Artin extends to all groups so the above is a case of independence of characters.

Artin does not mention pointwise independence.

Dedekind §164–165 invents the idea of linear independence (not for the first time, or the last) right before our eyes –

Dedekind §164–165 invents the idea of linear independence (not for the first time, or the last) right before our eyes – casting about for the right motivations, the right definitions, the right terms to express them, the right theorems.

Looking ahead

Looking ahead

Artin, esp. with Tate, would take this into class field theory and Galois cohomology.

Looking ahead

Artin, esp. with Tate, would take this into class field theory and Galois cohomology.

Serre and Grothendieck restore the link with monodromy and Riemann surfaces, in isotrivial and étale covers, cohomology, and fundamental groups.